Levinson's theorem in one dimension: heuristics

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Abstract. For partial waves in three dimensions, Levinson's theorem asserts that $\delta(0) = \pi n_b$, where $\delta(p)$ is the phase shift at wavenumber p, and n_b the number of bound states ($\delta(\infty) = 0$ by convention). The corresponding theorem in one dimension calls first for a systematic parametrisation of the transmission amplitude $T(p) = \cos \theta \, e^{i\tau}$, and of the left (right)-incidence reflection amplitudes $R_{L,R}(p) = i \sin \theta \exp(i\tau \pm i\rho)$, where the phase angles τ, θ, ρ are functions of p. If the potential is not everywhere zero, and excluding throughout the exceptional case where it has a zero-energy bound state, heuristic arguments show that $R_{L,R}(0) = -1$, that $\theta(0) = \frac{1}{2}\pi$ ($-\frac{1}{2}\pi$) when n_b is odd (even), and that $\tau(0) = \pi(n_b - \frac{1}{2})$; (by convention, $\tau(\infty) = 0 = \theta(\infty)$). Thus $|\tau(0)|$ cannot be less that $\frac{1}{2}\pi$, no matter how weak the potential; the transition to the limit of zero potential is non-uniform. In the special case of reflection-symmetric potentials, $\rho = 0$, one can subdivide $n_b = n_b^{(e)} + n_b^{(o)}$, and define even- and odd-parity phase shifts $E = \frac{1}{2}(\tau + \theta)$ and $\Delta = \frac{1}{2}(\tau - \theta)$; then $E(0) = \pi(n_b^{(e)} - \frac{1}{2})$, $\Delta(0) = -\pi n_b^{(o)}$. The appendix shows how $E(0)$ is obtainable by suitably adapting the familiar s-wave argument which exploits the analyticity properties of the Jost solutions.

1. Introduction

The overall behaviour of transmission and reflection amplitudes in one dimension (1D) seems to have been less studied, except in the WKB approximation, than that of the partial-wave amplitudes in 3D, about which there can be few simplicities left to discover (see e.g. Calogero 1967, Newton 1982). In any case, several pieces of quite basic information are not obtainable in practice from textbooks or monographs. For instance, recent calculations of the Casimir effect for scalar fields (Aoyama 1984) and for the Proca field (Barton and Dombey 1984) effectively require, implicitly or explicitly, the previously unavailable 1D version of Levinson's theorem (Levinson 1949). Here we aim, first, to formulate this problem, which involves generalising the familiar 3D phase-shift parametrisation to 1D scattering, where it is more elaborate rather than less. Second, we aim to suggest an answer by a preliminary and heuristic argument whose basic strategy is outlined later in this section. (A standard mathematical treatment, though only under fairly restrictive conditions on the potential, has been devised by Plaskett (1984).) In particular, we shall not try here to exploit nor even to formulate the analyticity properties of the 1D amplitudes as functions of wavenumber p or of energy p^2, except briefly in the appendix for the special case of reflection-symmetric potentials, where the results are obtainable by a fairly obvious adaptation of the partial-wave Jost solutions. Meanwhile, the reader might envisage a real local

potential $U(x)$, vanishing exactly† for say $x \leq x_1$ and $x \geq x_2$, and piecewise continuous or at least integrable.

We write the wave equation as

$$-d^2\psi/dx^2 + U(x)\psi = p^2\psi. \qquad (1.1)$$

The scattering states are those with a continuous positive energy spectrum $p^2 \geq 0$. The bound states, with square-integrable wavefunctions, have a discrete negative-energy spectrum with $p^2 \equiv -b^2 < 0$. The number of bound states will be written as n_b. Free particles satisfy the equation without the term in U.

As we shall see, the feature characteristic of 1D is that both of the linearly independent solutions of (1.1) must be considered, whereas the 3D reduced radial wavefunction $\psi(r) = rR(r)$ is uniquely defined from the outset by the p-independent boundary condition $\psi(0) = 0$. Recall that in the 3D s-wave, for instance, the phase shift $\delta(p)$ is linked to the wavefunction ψ_s, the S-matrix element S_s and the scattering amplitude f_s by $\psi_s(r \to \infty) \sim \text{constant} \times \sin(pr + \delta)$, $S_s = \exp(2i\delta)$ and $f_s = \exp(i\delta)\sin(\delta)/p$. Then one can show that $\delta(\infty) = 0 \bmod \pi$, and $\delta(0) = 0 \bmod \pi$; with the standard convention that $\delta(\infty) = 0$, Levinson's theorem asserts that $\delta(0) = \pi n_b$, where n_b is the number of s-wave bound states.

In § 2 we define the 1D transmission and reflection amplitudes; determine to what universal constraints they are subject at threshold ($p = 0$), or for reasons of symmetry; and in (2.13), (2.14) parametrise them compatibly with these constraints in terms of three angles, namely $\tau(p)$, the phase of the transmission amplitude; $\theta(p)$, which basically determines the ratio of reflection to transmission; and $\rho(p)$, which determines the phase difference between reflection from the left and from the right. The main conclusions‡ and § 2 are (2.10) and (2.12), which in terms of our parametrisation becomes (2.17), (2.18). Only at this stage can one formulate Levinson's problem, which is to connect $\tau(0)$ and $\theta(0)$ with n_b.

Section 3 considers the special case of symmetric potentials ($U(x) = U(-x)$), where the solutions of (1.1) can be chosen to have even or odd parity. One can deal with these separately, by defining an even phase shift $E = \frac{1}{2}(\tau + \theta)$ and an odd phase shift $\Delta = \frac{1}{2}(\tau - \theta)$; then $E(0)$ is linked to $n_b^{(e)}$ and $\Delta(0)$ to $n_b^{(o)}$, the numbers of even-parity and of odd-parity bound states respectively (so that $n_b = n_b^{(e)} + n_b^{(o)}$). The main results of § 3 are equations (3.3*b*) and (3.6*b*). It is remarkable that $|E(0)|$ can never be less than $\frac{1}{2}\pi$, however weak the potential, in sharp contrast to the 3D result quoted above.

When re-expressed in the form (3.10), these results suggest a generalisation which § 4 eventually shows to be correct even for asymmetric potentials, and which accordingly constitutes the sought-for 1D version of Levinson's theorem.

The reason why symmetric potentials are dealt with first is that this case is somewhat less cumbersome: it serves to display the basic strategy with fewer distractions, and allows a sharper focus on the crucial points of difference between 1D and 3D. Moreover it turns out in § 4 that the central counting problem in the general (asymmetric) case can be mapped onto that already solved in § 3, and need not be tackled again.

† In this paper we do not ask just how fast $U(|x| \to \infty)$ needs to decrease for the conclusions to remain valid. At first sight, the arguments most vulnerable to departures from this simplification are those leading to the threshold condition, equation (2.10), below.

‡ As far as the present writer can ascertain, the results (2.10) and (2.12) correctly derived are either new, which would be surprising, or deplorably under-publicised. Though (2.12) is asserted by Cohen-Tannoudji *et al* (1982), p 365, the present writer can see no force in the plausibility argument by which they support it, because the argument draws different conclusions about T and R simply from the unitarity condition (2.6) into which T and R enter on a perfectly equal footing. Their notation just happens to disguise this.

The basic idea is to proceed by the following stages:

(i) Discretise the positive-energy solutions of (1.1) by imposing fictitious boundary conditions

$$\psi(\pm L) = 0 \tag{1.2}$$

at distances L much greater than any length scale characteristic of the potential or of any of its bound states (thus $L \gg x_1, x_2, b^{-1}$), anticipating whenever convenient the limit $L \to \infty$ which is understood to be taken at the end. (Bound states, unless there is one at zero energy, are not affected by (1.2), since their wavefunctions decrease exponentially, like $\exp(-b|x|)$.)

(ii) Choose an allowed wavenumber P high enough that for $p \geq P$ the effects of the potential are negligible, and the interacting wavefunctions indistinguishable from those for free particles. (Such a choice will be possible provided the potential is not too singular and provided it falls fast enough as $|x| \to \infty$.) This coincidence has two crucial but equivalent consequences, and the argument can continue by focusing on either. We proceed first with the traditional and physically perhaps more intuitive approach, though eventually it may appear the weaker, and return to the other alternative below. Here, one notes that the coincidence between the wavefunctions for $p \geq P$ entails in particular that, in the light of (2.15), we can set $\theta = 0 = \tau$, and equivalently $E = 0 = \Delta$. Thus, for $p \geq P$, there is a natural one-to-one correspondence between the solutions of (1.1) and those of the free-particle equation.

(iii) Count how many solutions there are of (1.1) with $0 \leq p \leq P$; let this number be $\mathcal{N}$, and let the number of such solutions of the free equation be $\mathcal{N}_0$.

(iv) Argue that the total number of solutions (positive and negative energy) does not depend on the presence or absence of the potential, which one can imagine switching on or off adiabatically; equating the total number of solutions with and without interaction gives

$$\mathcal{N}_0 = \mathcal{N} + n_b, \tag{1.3}$$

which then delivers the desired result.

One should note that the choice both of L and of P obviously depends on the potential $U(x)$. Having made these choices for given U, the adiabatic switching envisaged at stage (iv) consists in replacing $U(x)$ by $sU(x)$, and then varying s continuously between 0 and 1; the requisite constancy of the number of solutions emerges as a plausible concomitant of such variation. (Admittedly, if s were to be increased above 1, then the original choices of L and P might need to be revised.)

To trace this approach to its origins appears to be difficult. The present writer first met it in lectures by D M Brink at Oxford in 1961–62. Weinberg (1965) uses it but attributes it, undated and unreferenced, to J Schwinger. Martin (1958) uses it in the much harder case of non-local potentials; more important from our point of view, he discusses carefully the conditions under which the number of solutions (i.e. the dimensionalities of the respective vector spaces) are indeed the same, and shows in passing that these conditions are satisfied by not-too-singular local potentials of the type considered here. Thus he provides the most thorough justification available for the traditional approach to (1.3).

The alternative approach focuses simply on the number of nodes of the discretised solutions. The free solution for $p = P$, and hence the assumed coincident interacting solution, has a number of nodes visible by inspection, as will be obvious automatically in §§ 3 and 4 below. But standard Sturm–Liouville theory tells us that in both sequences

of discretised energy eigenstates, interacting or free, each state has just one more node than the state below it in the same sequence, and that in both sequences the ground state is nodeless (except for the end points at $x = \pm L$). Therefore the total number of interacting solutions with $p \leq P$ is automatically the same as the total number of free solutions with $p \leq P$, the members of either sequence being counted by the number of nodes in the state with $p = P$, common to both sequences. Thus the crucial equality (1.3) is validated once again. Though in the present writer's view this second argument is stronger, the first argument is not only traditional but probably keeps closer to the underlying physics. Hence from here on we shall revert to the language of the latter.

At the end of § 3 the results for $E(0)$ and $\Delta(0)$ are re-expressed in a form which suggests a generalisation to arbitrary (asymmetric) potentials: in particular they imply $\tau(0) = \pi(n_b - \frac{1}{2})$. This generalisation is then made in § 4, which also contains some final comments. The appendix, rather disjoint from the rest of the paper, sketches some analyticity arguments for symmetric potentials, illustrating from another angle how the characteristic difference between the 3D and 1D odd-parity case on the one hand, and the 1D even-parity case on the other hand, can be ascribed to the different boundary conditions at $x = 0$. Note that apart from some asides we exclude the exceptional case of potentials with an exactly zero-energy bound (i.e. discrete) state.

2. Threshold behaviour and parametrisation in one dimension

In 1D scattering one must distinguish between incidence from the left and from the right; the corresponding solutions of (1.1) are represented by ψ_L and ψ_R respectively†:

$$\psi_L(x \to -\infty) = e^{ipx} + R_L e^{-ipx}, \qquad \psi_L(x \to +\infty) = T_L e^{ipx} \qquad (2.1a, b)$$

$$\psi_R(x \to -\infty) = T_R e^{-ipx}, \qquad \psi_R(x \to +\infty) = e^{-ipx} + R_R e^{ipx}. \qquad (2.2a, b)$$

To define these functions and the amplitudes T, R uniquely, we adopt the convention

$$p \geq 0; \qquad (2.3)$$

in §§ 3 and 4 it will prove essential to adhere to (2.3) with absolute consistency.

Some very general constraints on the T's and R's emerge if one considers the Wronskians $W(a, b) \equiv ab' - a'b$, with a, b different solutions of (1.1) for the same p^2, and if one then exploits the fact that W is independent of x. Considering $W(\psi_L, \psi_R)$ and equating its values as $x \to \pm\infty$, one finds

$$T_L = T_R \equiv T, \qquad (2.4)$$

whence we drop the suffix from T. Since we deal only with real U (i.e. U is invariant under time-reversal), ψ_R^* say also solves (1.1), and similarly considering $W(\psi_L, \psi_R^*)$ one finds

$$R_L / R_R^* = -T/T^*. \qquad (2.5)$$

Unitarity, i.e. the conservation of flux, imposes

$$|T|^2 + |R_L|^2 = 1 = |T|^2 + |R_R|^2. \qquad (2.6)$$

Finally, if U is symmetric, then $\psi_R(-x)$ say is yet another solution, and consideration

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† It will appear almost immediately that $T_L = T_R$, and the suffixes on the transmission amplitude T will then be dropped.

of $W(\psi_L(x), \psi_R(-x))$ yields

$$R_L = R_R \qquad \text{(if } U(x) = U(-x)\text{)}. \tag{2.7}$$

Up to this point we have merely summarised common knowledge: see for instance Messiah (1964), Merzbacher (1970), Fluegge (1974), and especially Cohen-Tannoudji *et al* (1982).

The amplitudes $R_{L,R}$, though not T, depend on where the origin is chosen. Under a translation $x \to x' = x - a$, the definitions (2.1, 2) entail $T \to T' = T$, $R_L \to R'_L = \exp(2ipa)R_L$, and $R_R \to R'_R = \exp(-2ipa)R_R$. Thus even a potential with a centre of inversion escapes the constraint (2.7) unless this centre is chosen as the origin:

$$R_L/R_R = 1 \Rightarrow R'_L/R'_R = \exp(4ipa). \tag{2.8}$$

Asymptotically, as $p \to \infty$, we shall assume on physical grounds that the potential, being non-singular, becomes wholly transparent:

$$T(p \to \infty) = 1, \qquad R_L(p \to \infty) = 0 = R_R(p \to \infty). \tag{2.9}$$

At threshold, as $p \to 0$, we argue that the general rule is

$$R_L(p \to 0) = -1 = R_R(p \to 0), \qquad T(p \to 0) = 0. \tag{2.10}$$

To see this, consider say ψ_L, identified as that solution of (1.1) for which $\psi'_L/\psi_L = ip$ to the far right, and notice that in the interior region, once p^2 has fallen far below the lowest energy scale characteristic of $U(x)$, the equation itself becomes essentially independent of p. (This is just the situation familiar in 3D from the standard theory of the scattering length.) When the solution ψ_L is continued to x_1 on the left, its logarithmic derivative there will be given by some function $A(p)$, and in such a situation there is no general reason why $A(p = 0)$ should assume one value rather than another. Thus, the generic case, to which our argument applies, is $A(0) \neq 0$; in other words this holds for almost all potentials. Then by continuity to the left we have from (2.1a) that

$$A(0) = \lim_{p \to 0} ip(e^{ipx_1} - R_L e^{-ipx_1})/(e^{ipx_1} + R_L e^{-ipx_1}). \tag{2.11}$$

But in view of the explicit factor p in the numerator, $A(0) \neq 0$ implies $R_L \to -1$, whence $T = 0$ follows by unitarity (2.6). A similar argument applies to R_R. The exceptional case[†] $A(0) = 0$, where (2.10) fails, and which in this paper we ignore, arises for potentials that have a discrete state at exactly zero energy.

In fact zero is a very special value of T, not realisable for any non-zero real value of p:

$$T(p \neq 0) \neq 0. \tag{2.12}$$

This is shown by *reductio ad absurdum*. If $T = 0$ for $p > 0$ (when the asymptotic solutions must be linear combinations of $\exp(\pm ipx)$), then by (2.1b) ψ_L is identically zero for all $\psi \geq x_2$; since ψ_L solves the differential equation (1.1), this in turn implies that ψ_L is zero for all x, contrary to the assumption that to the far left it is given by (2.1a). Of course (2.12) jointly with unitarity implies that for $p \neq 0$ there can never be total reflection: $|R_{L,R}(p \neq 0)| < 1$.

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† We have found no plausible general arguments applicable in this case along the lines of the present paper. Some limited conclusions can be reached by exploiting analyticity properties: see the remarks at the end of § 3.

It should be stressed that in principle the transmission and reflection amplitudes are measurable, both in magnitude and in phase; one need only split the incident beam, and then observe the interference which results when the incident, transmitted or reflected beams are recombined in pairs. We adopt the convention that phases, as well as amplitudes, change smoothly with p; then everything is uniquely defined up to just one integer multiple of 2π which could still be added simultaneously to the phases of T, R_L and R_R, and which is fixed by one further conventional choice made in equation (2.15) below.

To parametrise T and $R_{L,R}$ conveniently and without loss of generality, we first accommodate the constraints (2.4-7) by writing

$$T = \cos\theta\, e^{i\tau}, \tag{2.13}$$

$$R_L = i\sin\theta\, e^{i\tau + i\rho}, \tag{2.14a}$$

$$R_R = i\sin\theta\, e^{i\tau - i\rho}, \tag{2.14b}$$

with real θ, τ, ρ all smooth functions of p, by the convention just explained. For symmetric potentials subject to (2.7),

$$\rho = 0 \qquad (\text{if } U(x) = U(-x)). \tag{2.14c}$$

Some care is needed to make the phase angles τ, θ, ρ unique while adapting them to the asymptotic and threshold properties established above. Compatibly with (2.9) we adopt the convention

$$\theta(\infty) = 0 = \tau(\infty). \tag{2.15}$$

By continuity in p, this determines $\tau(p)$ uniquely through (2.13); but in view of (2.14a, b) it determines θ only to within a sign until we know more about ρ, because these expressions are invariant under the simultaneous replacements $\theta \to -\theta$ and $\rho \to \rho + \pi$. Since $R_L/R_R = \exp(2i\rho)$, and since, by (2.10), $R_L/R_R = 1$ at $p = 0$, we have $\exp(2i\rho(0)) = 1$, and adopt the further convention

$$\rho(0) = 0, \tag{2.16}$$

which allows one to accommodate (2.14c) if appropriate. Then R_L/R_R determines $\rho(p)$ by continuity for all p. Notice that we are not free to assume that $\rho(\infty)$ vanishes (i.e. this is not a matter of convention); for instance, even if $U(x) = U(-x)$ so that $\rho = 0$, yet the discussion relating to (2.8) shows that the same potential shifted through a distance a gives $R'_L/R'_R = \exp(4ipa) = \exp(2i\rho')$, so that ρ' fails to approach any limit as $p \to \infty$.

With ρ thus fixed by (2.16), R_L or R_R in (2.14a, b) now do determine the sign of θ, and all the parameters are specified uniquely. Equation (2.12) gives the strict inequalities

$$-\tfrac{1}{2}\pi < \theta(p) < \tfrac{1}{2}\pi, \qquad \text{for } p > 0, \tag{2.17}$$

and the threshold conditions (2.10) give

$$\theta(0) = \pm\tfrac{1}{2}\pi, \qquad \tau(0) = \pi(2\nu \pm \tfrac{1}{2}), \tag{2.18a, b}$$

where ν is an integer. The signs in (2.18a) and (2.18b) must be the same, but we do not yet know how to determine them. In §§ 3 and 4, alternative signs correspond to the alternatives in (2.18).

The question addressed in the present paper is to determine this sign, and to determine the integer ν, or equivalently the phase $\tau(0)$, in terms of information about the bound states of the potential. The end result is given in (3.10).

To appreciate the status of the amplitudes T and R it may help to recognise them as elements of a 2×2 unitary scattering matrix $S(p)$, acting on two-component vectors (C_L, C_R) whose entries are the coefficients in the expansion of an arbitrary positive-energy solution in the form $\psi = (C_L \psi_L + C_R \psi_R)$. In fact $S(p)$ is the reduced S-matrix, and we define for instance $\langle \psi_{L,p}^{(out)} | \psi_{R,p'}^{(in)} \rangle = 2\pi\delta(p - p') S_{LR}(p)$, where the (in) states are just those displayed (though without the superfix) in equations (2.1, 2) while the (out) states are given by

$$\psi_L^{(out)} = \psi_R^{(in)*}, \qquad \psi_R^{(out)} = \psi_L^{(in)*}. \tag{2.19}$$

This is the natural definition under which S becomes the unit matrix when an incident wavepacket is simply transmitted without change of amplitude or phase. Accordingly,

$$S(p) = \begin{pmatrix} S_{LL}, & S_{LR} \\ S_{RL}, & S_{RR} \end{pmatrix} = \begin{pmatrix} T, & R_R \\ R_L, & T \end{pmatrix}$$

$$= e^{i\tau} \begin{pmatrix} \cos\theta, & ie^{-i\rho}\sin\theta \\ ie^{i\rho}\sin\theta, & \cos\theta \end{pmatrix}, \tag{2.20}$$

$$S(\infty) = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}, \qquad S(0) = \begin{pmatrix} 0, & -1 \\ -1, & 0 \end{pmatrix}. \tag{2.21a, b}$$

For later reference we record the eigenvalues $S^{(1,2)}$ and the corresponding eigenfunctions $\psi^{(1,2)}$ of the S-matrix. One finds straightforwardly

$$S^{(1)} = \exp i(\tau + \theta), \qquad S^{(2)} = \exp i(\tau - \theta), \tag{2.22}$$

and, apart from irrelevant norming constants,

$$(C_L^{(1,2)}, C_R^{(1,2)}) = (e^{-i\rho/2}, \pm e^{i\rho/2}), \tag{2.23}$$

with the upper sign appropriate to $S^{(1)}$; or in other words, in view of (2.2, 3),

$$\psi^{(1)}(x \to \pm\infty) = e^{i(\tau+\theta)/2} \cos[px + \tfrac{1}{2}(\pm\tau \pm \theta - \rho)], \tag{2.24}$$

$$\psi^{(2)}(x \to \pm\infty) = e^{i(\tau-\theta)/2} \sin[px + \tfrac{1}{2}(\pm\tau \mp \theta - \rho)]. \tag{2.25}$$

Note that $\psi^{(1,2)}$ are essentially real functions of x. In the special case of symmetric potentials, when $\rho = 0$, $\psi^{(1)}$ and $\psi^{(2)}$ become, respectively, the even- and odd-parity eigenfunctions of S; hence they automatically become the unique even- and odd-parity eigenfunctions of the Hamiltonian which we then label as $\psi^{(e)}$ and $\psi^{(o)}$. Dropping the irrelevant prefactors in (2.24, 25), we define even- and odd-parity phase shifts E and Δ and write

$$\psi^{(e)}(x \to \pm\infty) = \cos(px \pm E), \tag{2.26}$$

$$\psi^{(o)}(x \to \pm\infty) = \sin(px \pm \Delta), \tag{2.27}$$

where, using (2.18),

$$E(p) = \tfrac{1}{2}(\tau + \theta), \qquad E(\infty) = 0, \qquad E(0) = \pi(\nu \pm \tfrac{1}{2}), \tag{2.28a, b, c}$$

$$\Delta(p) = \tfrac{1}{2}(\tau - \theta), \qquad \Delta(\infty) = 0, \qquad \Delta(0) = \pi\nu. \tag{2.29a, b, c}$$

3. The special case of symmetric potentials

The two linearly independent solutions of (1.1) in a symmetric potential may be chosen to be even and odd, behaving asymptotically as given by equations (2.26-29). To each in turn we now apply the discretisation and counting procedure outlined towards the end of § 1.

3.1. Even-parity solutions

On free solutions, having $E = 0$ in (2.26), the fictitious boundary conditions (1.2) impose the constraint $p_n L = \pi(n + \frac{1}{2})$, with $n = 0, 1, 2, \ldots$. We choose $P = \pi(N + \frac{1}{2})/L$. Then the number of free solutions with $0 \leq p \leq P$ is $\mathcal{N}_0^{(e)} = (N + 1)$.

On interacting solutions (2.26), the fictitious boundary conditions impose

$$p_n L + E(p_n) = \pi(n + \frac{1}{2}). \tag{3.1}$$

We must determine the lowest possible values $n_{\min}^{(e)}$ of n and $p_{\min}^{(e)}$ of p. Then the number of interacting solutions with $0 \leq p \leq P$ will be $\mathcal{N}^{(e)} = (N - n_{\min}^{(e)} + 1)$, whence the basic rule (1.3) yields $n_b^{(e)} = n_{\min}^{(e)}$ for the number $n_b^{(e)}$ of even-parity bound states. Now (3.1) shows that, as $L \to \infty$, p_n with any finite n, hence $p_{\min}$ in particular, tends to zero. Therefore, while determining $n_{\min}^{(e)}$, we can in (3.1) replace $E(p_n) \to E(0) = \pi(\nu \pm \frac{1}{2})$, which leads to

$$p_{\min}^{(e)} L = \pi(n_{\min}^{(e)} + \frac{1}{2} - \nu \mp \frac{1}{2}). \tag{3.2}$$

Next, we must recall that $\psi^{(e)}(x)$ takes its asymptotic form (2.26) for all $x \geq x_2$; hence, as $L \to \infty$, $\psi^{(e)}$ is given by (2.26) not only at $x = L$, but for all x in the range $x_2 \leq x \leq L$. This implies that $p_{\min}$ cannot be zero, because over this stretch $\psi^{(e)}$ would then be independent of x, and would therefore vanish simply because it vanishes at $x = L$; but, being a solution of the differential equation (1.1), $\psi^{(e)}$ would then vanish everywhere. Accordingly, after discretisation, the basic convention (2.3) sharpens to the strict inequality $p > 0$, and (3.2) entails $n_{\min}^{(e)} = (\nu + \frac{1}{2} \pm \frac{1}{2})$, $p_{\min}^{(e)} = \pi/L$. (Contrast $p_{\min}^{(e)}$ with the lowest value $\pi/2L$ for free solutions.) Thus our end results may be written

$$\nu = n_b^{(e)} - \frac{1}{2} \mp \frac{1}{2}, \qquad E(0) = \pi(n_b^{(e)} - \frac{1}{2}). \tag{3.3a, b}$$

3.2. Odd-parity solutions

We abbreviate the argument, because it follows closely that for even parity; but the outcome is different, and the precise quantitative differences should be watched.

Free solutions, with $\Delta = 0$ in (2.27), satisfy $p_n L = \pi n$, $n = 1, 2, 3, \ldots$. Now we choose $P = \pi N$. Then the number of free solutions with $0 \leq p \leq P$ is $\mathcal{N}_0^{(o)} = N$.

Interacting solutions (2.27) obey

$$p_n L + \Delta(p_n) = \pi n. \tag{3.4}$$

With $n_{\min}^{(o)}$ and $p_{\min}^{(o)}$ the lowest allowed values, one has $\mathcal{N}^{(o)} = (N - n_{\min}^{(o)} + 1)$ and, for the number $n_b^{(o)}$ of odd-parity bound states, $n_b^{(o)} = \mathcal{N}_0^{(o)} - \mathcal{N}^{(o)} = (n_{\min}^{(o)} - 1)$. To determine the minimum values we must again replace $\Delta(p_{\min})$ by $\Delta(0) = \pi\nu$. Then

$$p_{\min}^{(o)} L = \pi(n_{\min}^{(o)} - \nu). \tag{3.5}$$

Again $p_{\min}^{(o)}$ must be strictly positive, whence $n_{\min}^{(o)} = (\nu + 1)$ and $p_{\min}^{(o)} = \pi/L$. (This time

$p_{\min}^{(o)}$ does coincide with the lowest free p-value; note also that $p_{\min}^{(o)} = p_{\min}^{(e)}$†.) Thus our end result is

$$\nu = n_b^{(o)}, \qquad \Delta(0) = \pi n_b^{(o)}. \qquad (3.6a, b)$$

3.3. Comments

The result (3.6b) could have been foreseen: odd-parity solutions automatically satisfy the boundary condition $\psi(0) = 0$, which is precisely the same as that for the reduced partial-wave functions in 3D. Since both the equation and the boundary conditions coincide, the result is naturally the same. By contrast, the even-parity solutions satisfy $\psi'(0) = 0$; the appendix sketches how this difference works itself through in arguments proceeding through the analyticity properties of the Jost solutions of equation (1.1), while § 3.4 provides an explicit illustration of the end-result (3.3b).

As they stand, equations (3.3) and (3.6) do not immediately suggest a generalisation to potentials that are not symmetric. To this end we first re-express them in terms of $\tau(0)$ and $\theta(0)$, according to (2.28a) and (2.29a), recalling also (2.18):

$$E(0)/\pi = (\tau(0) + \theta(0))/2\pi = n_b^{(e)} - \tfrac{1}{2} = \nu \pm \tfrac{1}{2}, \qquad (3.7)$$

$$\Delta(0)/\pi = (\tau(0) - \theta(0))/2\pi = n_b^{(o)} = \nu. \qquad (3.8)$$

For the total number of bound states this gives

$$n_b \equiv n_b^{(e)} + n_b^{(o)} = 2\nu + \tfrac{1}{2} \pm \tfrac{1}{2}. \qquad (3.9)$$

Evidently the upper (lower) signs apply when n_b is odd (even), which implies $n_b^{(o)} = n_b^{(e)} - 1$ (or $n_b^{(o)} = n_b^{(e)}$), since the states alternate in parity as the energy rises. Therefore the alternatives are the following:

n_b is odd:

$$\nu = \tfrac{1}{2}(n_b - 1), \qquad \theta(0) = \tfrac{1}{2}\pi, \qquad \tau(0) = \pi(n_b - \tfrac{1}{2}); \qquad (3.10a)$$

n_b is even:

$$\nu = \tfrac{1}{2}n_b, \qquad \theta(0) = -\tfrac{1}{2}\pi, \qquad \tau(0) = \pi(n_b - \tfrac{1}{2}). \qquad (3.10b)$$

In the light of (2.22), the result for $\tau(0)$, which is common to both alternatives, can evidently be expressed as

$$(2i)^{-1} \operatorname{Tr} \log S(0) = \pi(n_b - \tfrac{1}{2}). \qquad (3.10c)$$

Though established so far only for symmetric potentials, equations (3.10) are in a form which at least could make sense in general. The next section shows that in fact they are always true.

Note finally that from (3.10) one might be tempted to hazard a guess about what happens when there is a zero-energy bound state. Let $n_{b\downarrow}$ denote the number of bound states not counting the one at zero energy; if we regard the situation as a halfway house between $n_b = n_{b\downarrow}$ and $n = n_{b\downarrow} + 1$, then one might conjecture that $\tau(0) = \pi n_{b\downarrow}$. Plaskett (1984) has shown that this is indeed correct. On the other hand, though one might conjecture also that $\theta(0) = 0$ (i.e. that $|T| = 1$), and though some potentials including the square well do give this result, it is not a general rule.

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† Of course there is no exact degeneracy amongst discrete states in 1D; if in the course of the argument we had not replaced $E(p)$ and $\Delta(p)$ at $p \approx \pi/L$ by their exact threshold values $E(0)$ and $\Delta(0)$, then this equality would fail. What should be asserted is that it holds to within corrections of higher relative order in $1/L$.

3.4. Illustration: the square-well potential

While for odd phase shifts (3.6*b*) merely replicates the familiar 3D Levinson theorem (as pointed out above), the new 1D result (3.3*b*) is worth verifying explicitly, first because of the heuristic nature of our arguments, and second, because in physics very general conclusions, even if reached by more rigorous mathematics, are notoriously accident-prone when applied to actual cases. In fact (3.3*b*) can be checked straightforwardly if rather tediously for all the familiar exactly-soluble 1D cases (e.g. the exponential and sech² potentials). Here we confine ourselves to the attractive square-well potential, setting, in equation (1.1), $U(x) = 0$ for $|x| > a$, and $U(x) = -m^2$ for $|x| \le a$. We define $q^2 = p^2 + m^2$, and adopt the scaled variables (used also by Barton and Dombey 1984) $\xi = p/m$, $\eta = q/m$, $\lambda = am$. Then the number of even-parity bound states is given by a standard argument (Schiff 1968) as

$$n_b^{(e)} = 1 + [\lambda/\pi], \qquad (3.11)$$

where [...] denotes the integer part. (For simplicity we continue to exclude zero-energy bound states so that λ/π is not an integer.)

The continuum wavefunctions are $\cos qx$ inside the well and $\cos(px + E)$ outside, up to constant factors; the phase shift E is determined by making ψ'/ψ continuous across $x = a$, which gives

$$\xi \tan(\xi\lambda + E) = \eta \tan(\eta\lambda), \qquad (\eta^2 = \xi^2 + 1). \qquad (3.12a, b)$$

It is natural and convenient to compare E with the auxiliary phase shift $(q - p)a = (\eta - \xi)\lambda$; hence we define

$$E = (\eta - \xi)\lambda + \varepsilon, \qquad (3.13)$$

and after some rearrangement (3.12*a*) yields

$$\tan \varepsilon = (\eta - \xi) \tan(\eta\lambda)/[\xi + \eta \tan^2(\eta\lambda)]. \qquad (3.14)$$

Since $\xi \to \infty$ entails $(\eta - \xi) \to 0$, and in view of the convention $E(\infty) = 0$, equation (3.13) shows that $\varepsilon(\infty) = 0$. Further, for positive ξ the RHS of (3.14) cannot diverge, whence we have the strict inequality

$$-\tfrac{1}{2}\pi < \varepsilon < \tfrac{1}{2}\pi. \qquad (3.15)$$

But with $\xi = 0$ and $\eta = 1$, (3.14) gives $\tan \varepsilon(0) = \cot \lambda$, whence $\varepsilon(0) = (\tfrac{1}{2}\pi - \lambda + n\pi)$ with some integer n; in view of (3.15) one has $n = [\lambda/\pi]$, or in other words

$$\varepsilon(0) = \tfrac{1}{2}\pi - \lambda + \pi[\lambda/\pi]. \qquad (3.16)$$

Together with (3.13) this gives finally

$$E(0) = \lambda + (\tfrac{1}{2}\pi - \lambda + \pi[\lambda/\pi]) = \pi([\lambda/\pi] + \tfrac{1}{2}). \qquad (3.17)$$

Comparing (3.11) and (3.17) one then obtains the expected result

$$E(0) = \pi(n_b^{(e)} - \tfrac{1}{2}).$$

4. The general case: asymmetric potentials

We show that (3.10) remains true even if the potential is asymmetric. Essentially this is done by mapping the counting problem in this general case onto that already solved

in § 3. The characteristic extra complication is that two continuously variable para-
meters are now needed to specify the positive-energy solutions (e.g., in equation (4.1)
below, the phase χ in addition to p); in the absence of symmetry, both of these must
be determined by the fictitious discretisation. Since the S-matrix eigenfunctions
(2.24, 25) have only the single adjustable parameter p, they no longer serve. Instead,
a general positive-energy solution of (1.1) is now specified in terms of an initially
arbitrary left-hand phase angle χ:

$$\psi(x \to -\infty) = \sin(px + \chi). \tag{4.1}$$

To determine the asymptotic form of ψ as $x \to +\infty$, we express the sine in exponential
form, use (2.2) and its complex conjugate to continue $\exp(\mp ipx)$ across the potential
from left to right, and find straightforwardly

$$\psi(x \to +\infty) = (\cos \theta)^{-1}[\sin(px + \chi + \tau) - \sin \theta \cos(px - \chi - \rho)]. \tag{4.2}$$

The paradox that the RHS of (4.2) diverges as $p \to 0$ ($\cos \theta \to 0$) is only apparent; it
stems from the left-right asymmetry of our procedure, which satisfies $\psi(-L) = 0$ first,
by choosing χ as a function of p, and looks for allowed values of p afterwards. It
would be possible, but more cumbrous, to devise a procedure treating the two conditions
$\psi(\pm L) = 0$ on an equal footing. Of course the basic reason for the complication is just
that the threshold result $T(0) = 0$ (equation (2.10)) does decouple the far-left and
far-right regions in real scattering situations, as opposed to those discretised artificially
to facilitate our arguments. Admittedly, at zero energy the scattering wavefunctions are
somewhat peculiar too; the appendix comments on this a little further.

Next, the above solutions are discretised. The condition $\psi(-L) = 0$ enforces $\chi = pL$,
and $\psi(L) = 0$ then enforces

$$\sin(2pL + \tau) = \sin \theta \cos \rho. \tag{4.3}$$

Let us define a principal-value arcsine function $\phi(p)$ by

$$-\tfrac{1}{2}\pi \leqslant \phi(p) = \sin^{-1}(\sin \theta \cos \rho) \leqslant \tfrac{1}{2}\pi. \tag{4.4}$$

The essential point is that, irrespective of the behaviour of the parity-violating phase
angle $\rho(p)$, all the properties of $\phi(p)$ that are relevant to our problem coincide with
those of $\theta(p)$. Thus: (i) by virtue of (2.15), $\phi(\infty) = \theta(\infty) = 0$. (ii) By virtue of the
strict inequalities (2.17), equation (4.4) likewise is a strict inequality for all finite p.
(iii) Finally, by virtue of (2.16) and (2.18a), we have $\phi(0) = \theta(0) = \pm\tfrac{1}{2}\pi$.

Now the discretisation condition (4.3) admits two sequences of solutions, satisfying
respectively one or other of the two constraints

$$2pL + \tau = \pi(2n + 1) - \phi, \tag{4.5a}$$

$$2pL + \tau = \pi(2n) + \phi, \tag{4.5b}$$

with integer n. The free solutions are $\sin[\pi k(x + L)/2L]$, with $p = \pi k/2L$, $k = 1, 2, 3, \ldots$.

The basic idea is to subdivide solutions, both free and interacting, into two types,
type $(+)$ with an even and type $(-)$ with an odd number of nodes. This classification
plays the same simplifying role as did parity in § 3: both criteria are invariant as the
potential is switched on or off adiabatically; with rising energy, the states alternate
according to both; and the criteria become synonymous if the potential is symmetric.
Free solutions with k odd (even) belong to type $(+)$ (type $(-)$). For interacting

solutions, as p drops below the reference wavenumber P (where $\tau = 0 = \theta$), the inequality (4.4) ensures that (4.5a) continues to govern states of type $(+)$, and (4.5b) those of type $(-)$.

In the light of these observations, one can see that the problem of counting the solutions of (4.5a) reduces precisely to that of counting the even-parity solutions in a symmetric potential, as solved in § 3.1, and that the problem of counting the solutions of (4.5b) reduces similarly to that of counting odd-parity solutions, as solved in § 3.2. In the expressions of § 3 we need merely replace θ by ϕ; then (3.1) (where $E = \frac{1}{2}(\tau + \theta)$) goes over into (4.5$a$), and (3.4) (where $\Delta = \frac{1}{2}(\tau - \theta)$) into (4.5$b$), and the arguments coincide step by step, since they depend only on the values of $E(0)$ and $\Delta(0)$, and since we have just demonstrated that $\phi(0) = \theta(0)$. Accordingly, the conclusions are also the same, and equations (3.10) indeed remain valid for asymmetric potentials, as anticipated at the end of § 3. This is the principal result of the present paper.

Thus, the simple answer to the question of how Levinson's theorem generalises to 1D is the assertion that the phase $\tau(p)$ of the transmission amplitude satisfies

$$\tau(0) = \pi(n_b - \tfrac{1}{2}), \tag{4.6}$$

given the convention $\tau(\infty) = 0$. Unfortunately, the simplest statement about the phases of the reflection amplitudes cannot be quite as simple as (4.6). We have seen that these phases depend on the choice of origin; even for an intrinsically symmetric potential an unwise choice makes them undefinable as $p \to \infty$, and partly a matter of convention at $p = 0$, as in equation (2.16). The statements that one can make about them individually† are contained in the parametrisation (2.14), together with (3.10a, b). The parallel assertions about the magnitudes rather than the phases of T and $R_{\text{L,R}}$ are just the threshold theorems (2.10). All this applies provided the potential has no zero-energy bound state.

At first sight, one might be startled by the conclusion that $|\tau(0)|$ can never be less than $\frac{1}{2}\pi$, no matter how weak the potential and no matter whether it is attractive or repulsive; or, for symmetric potentials, by the equivalent conclusion (3.3b) about the even-parity phase shift. In other words, as the potential strength approaches zero, $\tau(0)$ (like $T(0)$ and $R_{\text{L,R}}(0)$) assumes a limiting value different from that appropriate to free particles. Of course, in practice the consequences of these constraints are somewhat softened by the fact, readily seen from examples, that for a very weak potential the rise of $|\tau(p)|$ from 0 to $\frac{1}{2}\pi$ occurs in a very narrow range near threshold, where p^2 lies below the maximum value of $|U(x)|$. Nevertheless the result is simple, general, and perhaps unexpected enough to be pointed out. Possibly it can be regarded as a loose analogue, for continuum states, and for arbitrary potentials, of the famous result that in 1D any purely attractive potential, however weak, has at least one bound state.

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† The average of the reflection phases does satisfy a constraint which is at least independent of the choice of origin. Defining $\sigma = \frac{1}{2}(\arg R_{\text{L}} + \arg R_{\text{R}}) = \frac{1}{2}\pi + \arg(\sin \theta) + \tau$, the troublesome phase angle ρ cancels; observing the conventions (2.15) and the results (3.10a, b), we see that $\sigma(0) - \sigma(\infty)$ is given by $\tau(0)$ if n_b is odd, and by $\tau(0) - \pi$ if n_b is even.

Appendix. Arguments from analyticity for symmetric potentials

When the potential is symmetric, the analytic properties of the even- and odd-parity solutions, as functions of p, are obtainable from the standard treatment of 3D s-wave scattering in terms of the Jost solutions. Here we sketch this adaptation, following essentially the approach of Goldberger and Watson (1964); zero-energy bound states are again excluded for simplicity. The reader may find it entertaining to set the following argument step by step against the standard one. (A standard mathematical discussion, not confined to symmetric potentials, has been devised by Plaskett (1984).)

The standard case, covering the reduced s-wavefunction and the 1D odd-parity states, considers solutions $\psi^{(o)}(p, x)$ of (1.1) satisfying

$$\psi^{(o)}(p, 0) = 0, \qquad \psi^{(o)}(p, x \to \infty) = \sin(px + \Delta). \tag{A1}$$

To analyse $\psi^{(o)}$ one introduces an auxiliary ('regular') solution $\phi^{(o)}(p, x)$ defined by

$$\phi^{(o)}(p, 0) = 0, \qquad \phi_x^{(o)}(p, 0) = 1, \tag{A2a, b}$$

and the ('irregular') Jost solutions $f(\pm p, x)$ and their associated Jost functions $f(\pm p)$, defined by

$$f(\pm p, x \to \infty) = \exp(\mp i px), \tag{A3}$$

$$f(\pm p) \equiv f(\pm p, 0). \tag{A4}$$

For present use we define also

$$g(\pm p) \equiv f_x(\pm p, 0). \tag{A5}$$

In (A2) and (A5), suffixes denote partial derivatives; e.g. $\phi_x^{(o)} \equiv \partial \phi^{(o)}(p, x)/\partial x$. Of course the adjectives 'regular' and 'irregular' merely hark back to the 3D solution $R(r) = \psi(r)/r$; in 1D they imply no particular virtues or vices.

The s-wave and the 1D odd-parity problems, being mathematically identical, lead to the same theorem (3.6b). By contrast, in the 1D even-parity case (we omit superfixes (e)) one is concerned, instead, with solutions of (1.1) satisfying not (A1) but

$$\psi_x(p, 0) = 0, \qquad \psi(p, x \to \infty) = \cos(px + E). \tag{A6a, b}$$

These are analysed again by aid of the Jost solutions, and of another auxiliary solution defined by the conditions:

$$\phi(p, 0) = 1, \qquad \phi_x(p, 0) = 0. \tag{A7}$$

The conditions (A7), like (A2), are independent of p, and therefore make $\phi(p, x)$ analytic in the parameter p^2 which enters (1.1).

The auxiliary function ϕ can be expressed in terms of the Jost solutions:

$$\phi(p, x) = (2ip)^{-1}[-g(p)f(-p, x) + g(-p)f(p, x)]; \tag{A8}$$

the coefficients of $f(\pm p, x)$ on the right are obtained by enforcing the equality at $x = 0$ (where one exploits (A.6a) and (A7)), as $x \to \infty$ (where one exploits (A6b) and (A3)), and making use of the fact that the Wronskians formed with ϕ and $f(\pm p, x)$ are independent of x. (By contrast, the corresponding relation for odd parity reads

$$\phi^{(o)}(p, x) = (2ip)^{-1}[f(p)f(-p, x) - f(-p)f(p, x)].)$$

Since $\phi_x(p, 0)$ and $\psi_x(p, 0)$ both vanish, $\psi(x)$ must be some constant multiple of $\phi(x)$, say $\psi = \alpha\phi$. We compare their asymptotic behaviours, using (A8) and (A3) for ϕ and (A6*b*) for ψ:

$$\phi(p, x \to \infty) = (2ip)^{-1}[-g(p)e^{ipx} + g(-p)e^{-ipx}], \tag{A9}$$

$$\psi(p, x \to \infty) = \tfrac{1}{2}(e^{iE}e^{ipx} + e^{-iE}e^{-ipx}), \tag{A10}$$

and find, by equating the ratio of the coefficients of $e^{\pm ipx}$, that

$$g(p)/g(-p) = -\exp(2iE). \tag{A11}$$

One can also identify the enhancement factor

$$\alpha \equiv \psi(p, 0) = ipe^{-iE}/g(-p), \tag{A12}$$

which would be unity in the absence of any potential. Since, generically, $g(0) \neq 0$, we have the remarkable result

$$\alpha(0) = 0. \tag{A13}$$

Consequently $1/\alpha$ has a pole at $p = 0$, which proves crucial in the sequel. In fact (A13) is not totally unexpected. It is shown elsewhere (Barton 1983a, b) that the 1D enhancement factor vanishes at $p = 0$ both according to classical mechanics, and according to the expressions appropriate in the WKB regime of quantum mechanics, provided the potential has no turning points even for zero-energy incident particles. In both these cases one finds $\alpha(p) = p/(p^2 - U(0))^{1/2}$, though for non-singular potentials the WKB regime does not of course reach down to $p = 0$. Thus, equation (A13) merely extends the conclusion beyond the reach of the WKB approximation, and to potentials that may have turning points, barring only the exceptional case $g(0) = 0$, which again corresponds to a zero-energy bound state. But the zero-energy even-parity wavefunction is distinctly pathological: at the origin both ψ and ψ_x vanish, and *prima facie* this implies that ψ should vanish identically at all x. From the point of view of physical applications this is irrelevant: our results apply as the limit $p \to 0$ is approached.

The analyticity and reflection properties of $f(\pm p, x)$ and of $f(\pm p)$ are well known from the standard case. The functions $g(\pm p)$ share these properties, though not the asymptotic behaviour of $f(\pm p)$. In particular, $g(p)$ is analytic in the lower half of the complex p-plane. Moreover, its only zeros there lie on the negative imaginary axis, and correspond to (even-parity) bound states. One can see this from (A9), which shows that such zeros correspond to square-integrable solutions of (1.1); by Hermitecity, these belong to real negative eigenvalues $-p^2 = b^2 > 0$. The basic reflection property is $g^*(-p^*) = g(p)$. In particular, for real p we have

$$g^*(-p) = g(p), \tag{A14}$$

whence, writing

$$g(p) \equiv |g(p)| \exp(i\eta(p)), \tag{A15}$$

we have

$$|g(-p)| = |g(p)|, \qquad \eta(-p) = -\eta(p). \tag{A16}$$

But (A15, 16, 11) yield

$$g(p)/g(-p) = \exp(2i\eta(p)) = -\exp(2iE(p)), \tag{A17}$$

to be compared with $f(p)/f(-p) = +\exp(2iE)$.

Since by assumption the potential becomes ineffective as $p \to \infty$, $f(p, x)$ and $g(p)$ then approach the free-particle expressions for all x: $f(p, x) \to \exp(-\mathrm{i}px)$, $f_x(p, x) \to -\mathrm{i}p \exp(-\mathrm{i}px)$, whence in particular

$$g(p \to \infty) = -\mathrm{i}p. \tag{A18}$$

Accordingly, it proves convenient to define a function $h(p)$, evidently related to the enhancement factor:

$$h(p) \equiv g(p)/(-\mathrm{i}p), \qquad h(p \to \infty) \to 1. \tag{A19a, b}$$

The discussion relating to equations (A12, 13) above shows that $h(p)$ has a pole† at $p = 0$ (unrelated to bound states), and (A17) entails

$$h(p)/h(-p) = +\exp(2\mathrm{i}E(p)). \tag{A20}$$

Levinson's theorem now follows immediately from the usual integral counting the numbers of poles minus the number of zeros of the analytic function h:

$$\mathscr{I} \equiv \frac{1}{2\pi\mathrm{i}} \int_C \mathrm{d}z \frac{\mathrm{d}}{\mathrm{d}z} \log h(z), \tag{A21}$$

where the negative- (clockwise-) direction contour C runs along the real axis from $-\infty$ to $+\infty$, indented to a small semicircle of radius $\varepsilon \to 0$ below the origin, and is closed by a semicircle at infinity in the lower half-plane. Since the pole at the origin is excluded by the indentation, h is analytic and has only the bound-state zeros within C, whence

$$\mathscr{I} = -n_b^{(e)}. \tag{A22}$$

The integral can also be evaluated explicitly. In view of (A19b), there is no contribution from infinity. The contribution from the small semicircle is $-\frac{1}{2}$. Lastly, contribution from the rest of the real axis is evaluated by appeal to (A20):

$$\frac{1}{2\pi\mathrm{i}} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \mathrm{d}p \frac{\mathrm{d}}{\mathrm{d}p} \log h(p)$$

$$= \frac{1}{2\pi\mathrm{i}} \int_{\varepsilon}^{\infty} \mathrm{d}p \frac{\mathrm{d}}{\mathrm{d}p} \log[h(p) - h(-p)]$$

$$= \frac{1}{2\pi\mathrm{i}} \int_0^{\infty} \mathrm{d}p \frac{\mathrm{d}}{\mathrm{d}p} [2\mathrm{i}E(p)] = \frac{1}{\pi} [E(\infty) - E(0)]. \tag{A23}$$

Thus

$$\mathscr{I} = -n_b^{(e)} = -\tfrac{1}{2} + (1/\pi)[E(\infty) - E(0)], \tag{A24}$$

and the desired result (3.3b) follows.

Note added in proof. Contrary to the first sentence of the introduction, general studies of one-dimensional scattering do exist, in the more mathematical literature. They can be traced for instance from R G Newton 1983 *J. Math. Phys.* **24** 2152, and 1984 *J. Math. Phys.* **25** 2991. Moreover, the first-quoted paper proves Levinson's theorem (4.6) for the scattering process indicated in its title, which covers also the simpler case considered here of scattering by an ordinary potential. To the writer's regret he was, inexcusably, unaware of this work till after the revision of the present paper.

† This pole is present whenever there is a potential, no matter how weak, even though for free particles one naturally has $h(p) = 1$ for all p including $p = 0$. Evidently the residue at the pole must vanish with the strength of the potential.

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